Exercise 2.1

Show that basis functions $e^{\pm ikx}$ are orthogonal, i.e. that

$$\int_{-\pi}^{\pi} \mathrm{d}x e^{inx} e^{-imx} = 2\pi\delta_{n,m}, \quad \forall n.m \in \mathbb{Z}.$$

Solution:

Since $e^{inx}e^{-imx} = e^{i(n-m)x}$, if $n = m$, then

$$\int_{-\pi}^{\pi} \overbrace{e^{inx}e^{-imx}}^{=e^0=1} \,\mathrm{d}x = \int_{-\pi}^{\pi} \mathrm{d}x = 2\pi.$$

Now assume $n \neq m$. Then

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{inx}e^{-imx} \,\mathrm{d}x &= \int_{-\pi}^{\pi} e^{i(n-m)x} \,\mathrm{d}x \\
&= \left[\frac{e^{i(n-m)x}}{i(n-m)} \right]_{-\pi}^{\pi} \\
&= \frac{1}{i(n-m)} \left(e^{i\pi(n-m)} - e^{-i\pi(n-m)} \right) \\
&= \frac{1}{i(n-m)} \left(\left(e^{i\pi}\right)^{n-m} - \left(e^{-i\pi}\right)^{n-m} \right).
\end{aligned}$$

According to Euler's formula, $e^{i\pi} = -1$, so

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx} \,\mathrm{d}x = \frac{1}{i(n-m)} \left((-1)^{n-m} - (-1)^{n-m} \right) = 0.$$

Combining the two, we get the desired result

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx} \,\mathrm{d}x = 2\pi\delta_{n,m}.$$

3 points: the desired result was achieved with intermediate steps.

Exercise 2.2

Verify that the Fourier series in example (2.2) matches to the sine series in example 1.4 when $L = \pi$ using

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right).$$

Solution:

From Example 1.4,

$$x = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx)$$

and from Example 2.2, when $L = \pi$,

$$x = \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} \frac{i(-1)^k}{k} e^{ikx}.$$

From Euler's formula, it follows that

$$\sin(kx) = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right).$$

Therefore, by splitting and then recombining the series we get

$$\begin{aligned}
\sum_{\substack{k \neq 0 \\ k=-\infty}} \frac{i(-1)^k}{k} e^{ikx} &= \sum_{k=-\infty}^{-1} \frac{i(-1)^k}{k} e^{ikx} + \sum_{k=1}^{\infty} \frac{i(-1)^k}{k} e^{ikx} \\
&= \sum_{k=1}^{\infty} \frac{i(-1)^{-k}}{-k} e^{-ikx} + \sum_{k=1}^{\infty} \frac{i(-1)^k}{k} e^{ikx} \\
&= \sum_{k=1}^{\infty} \frac{i(-1)^k}{k} \left(e^{ikx} - e^{-ikx}\right) \\
&= \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \left(-\frac{i}{2}\right) \left(e^{ikx} - e^{-ikx}\right) \\
&= \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \cdot \frac{1}{2i} \left(e^{ikx} - e^{-ikx}\right) \\
&= \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx),
\end{aligned}$$

which is the sine series from Example 1.4, as expected.

3 points: the required result was shown with steps.

Exercise 2.3

Calculate the Fourier series in complex representation ($\sum_k c_k e^{ikx}$) for function

$$f(x) = \begin{cases} x + \pi, & \text{if } -\pi \leq x < 0 \\ -x, & \text{if } 0 \leq x \leq \pi \end{cases}$$

over the interval $x \in [-\pi, \pi]$.

Solution:

f clearly satisfies the Dirichlet condition, and thus the Fourier series converges to f on $[-\pi, \pi]$. Using Equation (36), we have for $k > 0$ that

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\pi}^{0} -x e^{-ikx} \, \mathrm{d}x + \frac{1}{2\pi} \int_{0}^{\pi} (x + \pi) e^{-ikx} \, \mathrm{d}x \\ &= -\frac{1}{2\pi} \left[\left(\frac{1}{k^2} + \frac{ix}{k} \right) e^{-ikx} \right]_{-\pi}^{0} + \frac{1}{2\pi} \left[\frac{1 + ik(x + \pi)}{k^2} e^{-ikx} \right]_{0}^{\pi} \\ &= \frac{e^{ik\pi} - ik\pi e^{ik\pi} - 1}{2\pi k^2} + \frac{e^{-ik\pi} \left(1 + 2ik\pi - e^{ikx}(1 + ik\pi) \right)}{2\pi k^2} \\ &= \frac{(-1)^k - ik\pi(-1)^k - 1 + (-1)^k + 2ik\pi(-1)^k - 1 - ik\pi}{2\pi k^2} \\ &= \frac{2(-1)^k + ik\pi(-1)^k - ik\pi - 2}{2\pi k^2} \\ &= \frac{(-1)^k (2 + ik\pi) - ik\pi - 2}{2\pi k^2}. \end{aligned}$$

For $k = 0$,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{0} -x \, \mathrm{d}x + \frac{1}{2\pi} \int_{0}^{\pi} (x + \pi) \, \mathrm{d}x = \pi.$$

Thus,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \pi + \frac{1}{2\pi} \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} \frac{(-1)^k (2 + ik\pi) - ik\pi - 2}{k^2}.$$

3 points: the result seems to be correct.

Exercise 2.4

Show that for real-valued coefficients a_k and b_k (cosine and sine basis) Parseval's theorem gives

$$\frac{1}{2L}\int_{-L}^{L} \mathrm{d}x f^2(x) = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Solution:

For the complex Fourier coefficients c_k, we have by Euler's identity $e^{ix} = \cos x + i \sin x$ that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |c_k|^2 &= \sum_{n=-\infty}^{\infty} \left| \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,\mathrm{d}x \right|^2 \\
&= \frac{1}{4}\sum_{n=-\infty}^{\infty} \left| \frac{1}{L}\int_{-L}^{L} f(x) \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{=\cos(-n\pi x/L)} \mathrm{d}x - \frac{i}{L}\int_{-L}^{L} f(x) \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{=-\sin(-n\pi x/L)} \mathrm{d}x \right|^2 .
\end{aligned}$$

The two integrals correspond to the cosine and sine coefficients a_k and b_k, therefore by splitting the series into two

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |c_k|^2 &= \frac{1}{4}\sum_{n=1}^{\infty} |a_n - ib_n|^2 + \frac{1}{4}\sum_{n=1}^{\infty} \left| \overbrace{a_n}^{=a_{-n}} + i \overbrace{b_n}^{=-b_{-n}} \right|^2 + \frac{1}{4}|a_0|^2 \\
&= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).
\end{aligned}$$

By Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |c_k|^2 = \frac{1}{2L}\int_{-L}^{L} f(x)^2\,\mathrm{d}x = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

3 points: the result is achieved, with steps.

Exercise 2.5

Let us consider an application of Fourier series: Consider an amplifier which for a pure sine input voltage $V_{\text{in}} = U_0 \sin(\omega t)$ gives the following output voltage

$$V_{\text{out}} = \begin{cases} -V_0 & \text{for } -T \leq t < 0 \\ V_0 & \text{for } 0 \leq t \leq T \end{cases}$$

where $\omega = \pi/T$. Compute the Fourier sine seires for V_{out} over period $[-T, T]$ (V_{out} is odd so cosine terms would drop out anyway). Use this to compute the harmonic distortion of the amplifier, defined as

$$D = \sqrt{\sum_{k=2}^{\infty} \frac{b_k^2}{b_1^2}}.$$

Solution:

The sine coefficients are, for $k > 0$,

$$\begin{aligned} b_k &= \frac{1}{T} \int_{-T}^{T} V_{\text{out}}(t) \sin\left(\frac{k\pi t}{T}\right) \mathrm{d}t \\ &= \frac{1}{T} \int_{-T}^{0} -V_0 \sin\left(\frac{k\pi t}{T}\right) \mathrm{d}t + \frac{1}{T} \int_{0}^{T} V_0 \sin\left(\frac{k\pi t}{T}\right) \mathrm{d}t \\ &= \frac{V_0}{T} \left[\frac{T}{k\pi} \cos\left(\frac{k\pi t}{T}\right)\right]_{-T}^{0} - \frac{V_0}{T} \left[\frac{T}{k\pi} \cos\left(\frac{k\pi t}{T}\right)\right]_{0}^{T} \\ &= \frac{V_0}{k\pi} (1 - \cos k\pi) - \frac{V_0}{k\pi} (\cos k\pi - 1) \\ &= \frac{2V_0}{k\pi} \left(1 - (-1)^k\right). \end{aligned}$$

Additionally,

$$a_0 = \frac{1}{T} \int_{-T}^{T} V_{\text{out}}(t) \, \mathrm{d}t = 0,$$

so the Fourier sine series becomes

$$V_{\text{out}} = \sum_{k=1}^{\infty} b_k \sin(k\omega t) = \frac{2V_0}{\pi} \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{k} \sin(k\omega t)$$

where the Dirichlet condition guarantees convergence. Thus, for the harmonic distortion,

$$
\begin{aligned}
D^2 = \sum_{k=2}^{\infty} \frac{b_k^2}{b_1^2} &= \sum_{k=2}^{\infty} \frac{\frac{4V_0^2}{k^2\pi^2}\left(1-(-1)^k\right)^2}{\frac{4V_0^2}{\pi^2}(1+1)^2} \\
&= \sum_{n=2}^{\infty} \frac{\left(1-(-1)^k\right)^2}{4k^2} = \frac{1}{2}\sum_{k=2}^{\infty} \frac{1-(-1)^k}{k^2} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1-(-1)^k}{k^2} - 1.
\end{aligned}
$$

From Exercise 1.5, we have that

$$
f(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \cos(kx) = x^2
$$

and that

$$
\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.
$$

By plugging $x = 0$, we get that

$$
f(0) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = 0 \iff \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}.
$$

Combining the two, we get that

$$
D^2 + 1 = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1-(-1)^k}{k^2} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = \frac{\pi^2}{12} + \frac{\pi^2}{24} = \frac{\pi^2}{8}
$$

and therefore

$$
D = \sqrt{\frac{\pi^2}{8} - 1}.
$$

3 points: the result seems to be correct, with steps shown.