Exercise 4.1

Use the properties of the Dirac delta function to compute the integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \delta\left(x^4 - 2\right) x^2.$$

Solution:

Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^4 - 2$. Clearly $g \in \mathcal{C}^1$. Since the real roots of $g(x)$ are $x_{1,2} = \pm 2^{1/4}$, by the change of variables (101) we get

$$\begin{aligned}\int_{-\infty}^{\infty} \delta\left(g(x)\right) x^2 \,\mathrm{d}x &= \sum_{i=1}^{2} \int_{-\infty}^{\infty} \delta\left(x - x_i\right) \frac{x^2}{|g'(x_i)|} \,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \delta\left(x + 2^{1/4}\right) \frac{x^2}{4 \cdot 2^{3/4}} \,\mathrm{d}x + \int_{-\infty}^{\infty} \delta\left(x - 2^{1/4}\right) \frac{x^2}{4 \cdot 2^{3/4}} \,\mathrm{d}x.\end{aligned}$$

Using the fundamental property of the Dirac delta function (96),

$$\int_{-\infty}^{\infty} \delta\left(x + 2^{1/4}\right) \frac{x^2}{4 \cdot 2^{3/4}} \,\mathrm{d}x + \int_{-\infty}^{\infty} \delta\left(x - 2^{1/4}\right) \frac{x^2}{4 \cdot 2^{3/4}} \,\mathrm{d}x = 2 \cdot \frac{2^{2/4}}{4 \cdot 2^{3/4}} = 2^{-5/4}.$$

3 points: the result should be correct by checking with Mathematica.

Exercise 4.2

Compute the Fourier transform for function

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-x/2}, & \text{for } x \geq 0 \end{cases}$$

and use the Parseval's theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+4x^2}.$$

Solution:

Calculating the Fourier transform, we get that

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{x\left(i\xi - \frac{1}{2}\right)}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{x\left(i\xi-\frac{1}{2}\right)}}{i\xi - \frac{1}{2}} \right]_0^{\infty} \\
&= -\frac{1}{\sqrt{2\pi}} \frac{1}{i\xi - \frac{1}{2}} \\
&= \sqrt{\frac{2}{\pi}} (1-2i\xi)^{-1}.
\end{aligned}$$

Since $\hat{f}^*(\xi) = \sqrt{\frac{2}{\pi}}(1+2i\xi)^{-1}$, we get

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+4x^2} = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1-2ix)(1+2ix)} = \frac{\pi}{2} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{f}^*(\xi)\,\mathrm{d}\xi.$$

By Parseval's theorem (104) we have that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+4x^2} &= \frac{\pi}{2} \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{f}^*(\xi)\,\mathrm{d}\xi = \frac{\pi}{2} \int_{-\infty}^{\infty} f(x) f^*(x)\,\mathrm{d}x \\
&= \frac{\pi}{2} \int_0^{\infty} e^{-x/2} e^{-x/2}\,\mathrm{d}x = \frac{\pi}{2} \left[-e^{-x} \right]_0^{\infty} \\
&= \frac{\pi}{2}.
\end{aligned}$$

3 points: the result should be correct.

Exercise 4.3

Compute the Fourier transform of function

$$f(x) = \begin{cases} \cosh(x) & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}.$$

It is useful to use the exponential representations for the hyperbolic sine and cosine

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x)\frac{e^x - e^{-x}}{2}.$$

Solution:

Using the exponential representation for $\cosh(x)$, the Fourier transform becomes

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\xi x} \cosh(x) \, \mathrm{d}x \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-a}^{a} e^{i\xi x} \left(e^x + e^{-x}\right) \, \mathrm{d}x \\
&= \frac{1}{2\sqrt{2\pi}} \left\{ \int_{-a}^{a} e^{x(i\xi+1)} \, \mathrm{d}x + \int_{-a}^{a} e^{x(i\xi-1)} \, \mathrm{d}x \right\} \\
&= \frac{1}{2\sqrt{2\pi}} \left\{ \left[\frac{e^{x(i\xi+1)}}{i\xi+1} \right]_{-a}^{a} + \left[\frac{e^{x(i\xi-1)}}{i\xi-1} \right]_{-a}^{a} \right\} \\
&= \frac{1}{2\sqrt{2\pi}} \left\{ \frac{e^{a(i\xi+1)} - e^{-a(i\xi+1)}}{i\xi+1} + \frac{e^{a(i\xi-1)} - e^{-a(i\xi-1)}}{i\xi-1} \right\}.
\end{aligned}$$

Converting the exponentials back to hyperbolic trig functions, we get

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \left\{ \frac{\sinh(ai\xi + a)}{i\xi + 1} + \frac{\sinh(ai\xi - a)}{i\xi - 1} \right\}.$$

Now, using addition formulas for sinh and cosh, we can expand the sums. Combining the two fractions and using the fact that cosh is even and sinh is odd, we get a simplified version (with the most tedious steps skipped)

$$\hat{f}(\xi) = -\frac{1}{\sqrt{2\pi}\,(\xi^2+1)} (2i\xi \sinh(ai\xi)\cosh(a) - 2\sinh(a)\cosh(ai\xi)).$$

Lastly, we can use the fact that $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cosh(x)$ to get

$$\hat{f}(\xi) = \frac{\sqrt{2/\pi}}{\xi^2+1} (\xi \sin(a\xi)\cosh(a) + \sinh(a)\cos(a\xi)).$$

3 points: the result seems to be correct.

Exercise 4.4

Show that the Fourier transform of convolution integral

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y\, f_1(y) f_2(x-y)$$

is the product of Fourier transforms of the component functions, i.e. that

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, h(x) e^{-ikx} = g_1(k) g_2(k)$$

where g_1 and g_2 are Fourier transforms of f_1 and f_2.

Solution:

Let's calculate the Fourier transform of h. With Fubini's theorem,

$$\begin{aligned}
\hat{h}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{i\xi x}\, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(y) f_2(x-y) e^{i\xi x}\, \mathrm{d}y\, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(y) f_2(x-y) e^{i\xi x}\, \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(y) \int_{-\infty}^{\infty} f_2(x-y) e^{i\xi x}\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

Now, setting $z = x - y$, we can do a change of variables to get, with the help from Fubini's theorem, that

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(y) \int_{-\infty}^{\infty} f_2(x-y) e^{i\xi x}\, \mathrm{d}x\, \mathrm{d}y &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(y) \int_{-\infty}^{\infty} f_2(z) e^{i\xi(y+z)}\, \mathrm{d}z\, \mathrm{d}y \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(y) e^{i\xi y} \int_{-\infty}^{\infty} f_2(z) e^{i\xi z}\, \mathrm{d}z\, \mathrm{d}y \\
&= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(y) e^{i\xi y}\, \mathrm{d}y \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(z) e^{i\xi z}\, \mathrm{d}z \right) \\
&= \hat{f}_1(\xi) \hat{f}_2(\xi).
\end{aligned}$$

Therefore $\mathcal{F}\{f_1 * f_2\} = \mathcal{F}\{f_1\}\mathcal{F}\{f_2\}$, where $\mathcal{F}$ denotes the Fourier transform.

3 points: the required result was reached with steps.

Exercise 4.5

Compute Fourier transform of function

$$f(x) = e^{-ax^2} \quad (a > 0).$$

Solution:

First, by completing the square $-ax^2 + i\xi x$ we get that

$$-ax^2 + i\xi x = \left(\sqrt{a}x - \frac{i\xi}{2\sqrt{a}}\right)^2 - \frac{\xi^2}{4a}.$$

Therefore by setting $\sigma \equiv -\xi/2\sqrt{a}$,

$$\int_{-\infty}^{\infty} e^{-ax^2+i\xi x}\,\mathrm{d}x = e^{-\xi^2/4a}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+i\sigma\right)^2}\,\mathrm{d}x.$$

Now set $u = \sqrt{a}x + i\sigma$, and thus $\mathrm{d}u = \sqrt{a}\,\mathrm{d}x$. Additionally,

$$\lim_{x\to\pm\infty} u(x) = \pm\infty + i\sigma.$$

With this change of variables, the integral becomes

$$e^{-\xi^2/4a}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+i\sigma\right)^2}\,\mathrm{d}x = \frac{e^{-\xi^2/4a}}{\sqrt{a}}\int_{-\infty+i\sigma}^{\infty+i\sigma} e^{-u^2}\,\mathrm{d}u,$$

which is the Gaussian integral with a known value of $\sqrt{\pi}$. The Fourier transform is therefore

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\xi x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{i\xi x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2+i\xi x}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}e^{-\xi/4a}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+i\sigma\right)^2}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\frac{e^{-\xi^2/4a}}{\sqrt{a}}\int_{-\infty+i\sigma}^{\infty+i\sigma} e^{-u^2}\,\mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}}\frac{e^{-\xi^2/4a}}{\sqrt{a}}\sqrt{\pi} \\
&= \frac{e^{-\xi^2/4a}}{\sqrt{2a}}.
\end{aligned}$$

3 points: the result should be correct, with steps.