Exercise 6.1

Compute Laplace transform of the convolution integral

$$f(x) = \int_0^x \mathrm{d}y\, e^{x-y} \sin(y).$$

Solution:

Noticing that

$$f(x) = \int_0^x e^{x-y} \sin(y)\, \mathrm{d}y = e^x * \sin(x),$$

then by Equation (154), we get that

$$\mathcal{L}\left\{f(x)\right\}(s) = \mathcal{L}\left\{e^x * \sin(x)\right\}(s) = \mathcal{L}\left\{e^x\right\}(s)\, \mathcal{L}\left\{\sin(x)\right\}(s).$$

From Examples 5.1 and 5.2,

$$\mathcal{L}\left\{e^x\right\}(s)\, \mathcal{L}\left\{\sin(x)\right\}(s) = \frac{1}{s-1} \cdot \frac{1}{s^2+1} = \frac{1}{(s^2+1)(s-1)}.$$

3 points: the result seems to be correct.

Exercise 6.2

Find the inverse Laplace transform for

$$g(s) = \frac{1}{(s+2)(s^2+5s+6)}$$

Apply again the previously computed transforms and the result (155) for product of two Laplace transforms.

Solution:

From the Laplace transforms table,

$$\mathcal{L}\left\{e^{-3x}\right\}(s) = \frac{1}{s+3} \quad \text{and} \quad \mathcal{L}\left\{xe^{-2x}\right\}(s) = \frac{1}{(s+2)^2}.$$

By (155) and integration by parts,

$$\begin{aligned}
\mathcal{L}^{-1}\{g(s)\}(x) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\cdot\frac{1}{s+3}\right\}(x) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{xe^{-2x}\right\}(s)\,\mathcal{L}\left\{e^{-3x}\right\}(s)\right\}(x) \\
&= \int_0^x ye^{-2y}e^{-3(x-y)}\,\mathrm{d}y = e^{-3x}\int ye^{y}\,\mathrm{d}y = e^{-3x}\left(\left[ye^{y}\right]_0^x - \int_0^x e^{y}\,\mathrm{d}y\right) \\
&= e^{-3x}\left(1 + xe^{x} - e^{x}\right).
\end{aligned}$$

3 points: the result looks correct.

Exercise 6.3

Find the solution for differential equation

$$y''(x) + 5y'(x) + 6y(x) = e^{-2x}$$

with initial conditions $y(0) = 0$, $y'(0) = 0$ using Laplace transform.

Solution:

Since the Laplace transform is linear, taking the Laplace transform of the left-hand side of the DE and using the derivative rule yields

$$\begin{aligned}\mathcal{L}\left\{y'' + 5y' + 6y\right\}(s) &= s\mathcal{L}\left\{y'\right\}(s) - y'(0) + 5s\mathcal{L}\left\{y\right\}(s) - 5y(0) + 6\mathcal{L}\left\{y\right\}(s)\\ &= s^2\mathcal{L}\left\{y\right\}(s) - sy(0) + 5s\mathcal{L}\left\{y\right\}(s) + 6\mathcal{L}\left\{y\right\}(s)\\ &= s^2\mathcal{L}\left\{y\right\}(s) + 5s\mathcal{L}\left\{y\right\}(s) + 6\mathcal{L}\left\{y\right\}(s).\end{aligned}$$

Similarly, the Laplace transform of the right-hand side of the DE gives

$$\mathcal{L}\left\{e^{-2x}\right\}(s) = \frac{1}{s+2}.$$

Equating the two and solving for $\mathcal{L}\left\{y\right\}(s)$ gives us

$$\mathcal{L}\left\{y\right\}(s) = \frac{1}{(s+2)(s^2+5s+6)}.$$

By the previous exercise,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+5s+6)}\right\}(x) = e^{-3x}\left(1 + xe^x - e^x\right).$$

Therefore

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+5s+6)}\right\}(x) = e^{-3x}\left(1 + xe^x - e^x\right),$$

which by differentiation is indeed the solution.

3 points: the result should be correct.

Exercise 6.4

Use Laplace transform to find the solution for differential equation

$$y''(x) + 4y(x) = \sin(2x)$$

corresponding to initial conditions $y(0) = 0$ and $y'(0) = -2$.

Solution:

The Laplace transform of the RHS is by linearity and the derivative rule

$$\begin{aligned}\mathcal{L}\left\{y'' + 4y\right\}(s) = s\mathcal{L}\left\{y'\right\}(s) - y'(0) + 4\mathcal{L}\left\{y\right\}(s) \\ &= s^2\mathcal{L}\left\{y\right\}(s) - sy(0) + 2 + 4\mathcal{L}\left\{y\right\}(s) \\ &= s^2\mathcal{L}\left\{y\right\}(s) + 4\mathcal{L}\left\{y\right\}(s) + 2\end{aligned}$$

and similarly, the Laplace transform of the LHS using known transforms becomes

$$\mathcal{L}\left\{\sin(2x)\right\}(s) = \frac{2}{s^2+4}.$$

We can now set the two equal, which gives

$$s^2\mathcal{L}\left\{y\right\}(s) + 4\mathcal{L}\left\{y\right\}(s) + 2 = \frac{2}{s^2+4}.$$

Solving for $\mathcal{L}\left\{y\right\}(s)$ gives

$$\mathcal{L}\left\{y\right\}(s) = \frac{2}{(s^2+4)^2} - \frac{2}{s^2+4}.$$

Now, by linearity and the convolution theorem,

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}-\frac{2}{s^2+4}\right\}(x) &= \mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{2}{s^2+4}\cdot\frac{2}{s^2+4}\right)\right\}(x)-\sin(2x) \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\sin(2x)\right\}(s)\,\mathcal{L}\left\{\sin(x)\right\}(s)\right\}(x)-\sin(2x) \\
&= \frac{1}{2}\int_0^x \sin(2y)\sin(2(x-y))\,\mathrm{d}y-\sin(2x) \\
&= \frac{1}{4}\int_0^x 2\sin\left(\frac{1}{2}(4y-2x+2x)\right)\sin\left(\frac{1}{2}(2x+2x-4y)\right)\mathrm{d}y-\sin(x) \\
&= \frac{1}{4}\int_0^x (\cos(4y-2x)-\cos(2x))\,\mathrm{d}y-\sin(2x) \\
&= \frac{1}{16}\int_{-2x}^{2x}\cos(u)\,\mathrm{d}u-\frac{1}{4}x\cos(2x)-\sin(2x) \\
&= \frac{1}{16}(\sin(2x)-\sin(-2x))-\frac{1}{4}x\cos(2x)-\sin(2x) \\
&= \frac{1}{8}\sin(2x)-\frac{1}{4}x\cos(2x)-\sin(2x) \\
&= -\frac{7}{8}\sin(2x)-\frac{1}{4}x\cos(2x).
\end{aligned}
$$

Therefore

$$y(x) = -\frac{7}{8}\sin(2x)-\frac{1}{4}x\cos(2x).$$

Plugging $y(x)$ back to the DE shows that it is indeed the solution.

3 points: the result should be correct.

Exercise 6.5

Use Laplace transform to solve differential equations

$$y''(x) + \omega^2 y(x) = a(1 - \theta(x - b))$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$ and $a, b > 0$. Verity this explicitly by inserting the obtained solution to the equation (which is useful to do every time).

Solution:

By linearity and the derivative rule, taking the Laplace transform of the LHS gives

$$\begin{aligned}
\mathcal{L}\left\{y'' + \omega^2 y\right\}(s) &= s\mathcal{L}\left\{y'\right\}(s) - y'(0) + \omega^2 \mathcal{L}\left\{y\right\}(s) \\
&= s^2 \mathcal{L}\left\{y\right\}(s) - sy(0) + \omega^2 \mathcal{L}\left\{y\right\}(s) \\
&= s^2 \mathcal{L}\left\{y\right\}(s) + \omega^2 \mathcal{L}\left\{y\right\}(s) \\
&= \mathcal{L}\left\{y\right\}(s)\left(s^2 + \omega^2\right)
\end{aligned}$$

and similarly, using known transforms, the Laplace transform of the RHS is

$$\begin{aligned}
\mathcal{L}\left\{a(1 - \theta(x - b))\right\}(s) &= a\left(\mathcal{L}\left\{1\right\}(s) - \mathcal{L}\left\{\theta(x - b)\right\}(s)\right) \\
&= a\left(\frac{1}{s} - \frac{e^{-sb}}{s}\right) \\
&= \frac{a}{s}\left(1 - e^{-sb}\right).
\end{aligned}$$

Setting the two transforms equal to each other and solving for $\mathcal{L}\left\{y\right\}(s)$ gives

$$\begin{aligned}
\mathcal{L}\left\{y\right\}(s) &= \frac{a\left(1 - e^{-sb}\right)}{s(s^2 + \omega^2)} \\
&= \frac{a}{s}\frac{1}{s^2 + \omega^2}\left(1 - e^{-sb}\right) \\
&= \frac{1}{\omega}\frac{a}{s}\frac{\omega}{s^2 + \omega^2}\left(1 - e^{-sb}\right) \\
&= \frac{1}{\omega}\frac{a}{s}\frac{\omega}{s^2 + \omega^2} - \frac{a}{\omega}\frac{\omega}{s^2 + \omega^2}\frac{e^{-sb}}{s}.
\end{aligned}$$

Therefore, by linearity, known transforms and the convolution theorem,

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{a\left(1-e^{-sb}\right)}{s(s^2+\omega^2)}\right\}(x) &= \frac{1}{\omega}\mathcal{L}^{-1}\left\{\mathcal{L}\left\{a\right\}(s)\,\mathcal{L}\left\{\sin(\omega x)\right\}(s)\right\}(x) - \frac{a}{\omega}\mathcal{L}^{-1}\left\{\mathcal{L}\left\{\sin(\omega x)\right\}(s)\,\mathcal{L}\left\{\theta(x-b)\right\}(s)\right\}(x) \\
&= \frac{1}{\omega}\int_0^x a\sin(\omega y)\,\mathrm{d}y - \frac{a}{\omega}\int_0^x \theta(x-b)\sin(\omega(x-y))\,\mathrm{d}y \\
&= \frac{a}{\omega^2}(1-\cos(\omega x)) + \frac{a}{\omega}\theta(x-b)\int_b^x \sin(\omega(y-x))\,\mathrm{d}y \\
&= \frac{a}{\omega^2}(1-\cos(\omega x)) - \frac{a}{\omega}\theta(x-b)\int_0^{b-x} \sin(\omega u)\,\mathrm{d}u \\
&= \frac{a}{\omega^2}(1-\cos(\omega x)) - \frac{a}{\omega^2}(1-\cos(\omega(b-x)))\theta(x-b).
\end{aligned}
$$

Therefore

$$
y(x) = \frac{a}{\omega^2}(1-\cos(\omega x) + (\cos(\omega(b-x))-1)\theta(x-b))
$$

and thus

$$
\begin{aligned}
y'(x) &= \frac{a}{\omega^2}(\omega\sin(\omega x) + \omega\sin(\omega(b-x))\theta(b-x) + \overbrace{\delta(x-b)(\cos(\omega(b-x))-1)}^{=0}) \\
&= \frac{a}{\omega^2}(\omega\sin(\omega x) + \omega\sin(\omega(b-x))\theta(x-b))
\end{aligned}
$$

and

$$
\begin{aligned}
y''(x) &= \frac{a}{\omega^2}(\omega^2\cos(\omega x) - \omega^2\cos(\omega(b-x))\theta(x-b) + \overbrace{\omega\sin(\omega(b-x))\delta(x-b)}^{=0}) \\
&= \frac{a}{\omega^2}(\omega^2\cos(\omega x) - \omega^2\cos(\omega(b-x))\theta(x-b)).
\end{aligned}
$$

Now

$$
\begin{aligned}
y'' + \omega^2 y &= a\cos(\omega x) - a\cos(\omega(b-x))\theta(x-b) + a - a\cos(\omega x) + a\cos(\omega(x-b))\theta(x-b) - a\theta(x-b) \\
&= a(1-\theta(x-b))
\end{aligned}
$$

as expected.

3 points: the desired result was reached with steps.